Hermite Polynomials

Hermite Differential Equations

The Differential Equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0,$$ where λ is a constant

is called Hermite's differential equation.

~~Hermi~~ Solution of Hermite's Equation :-

The Hermite Polynomial, denoted by $H_n(x)$, is the solution of the Hermite's differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0 \quad \text{——— (1)}$$

or $y'' - 2xy' + 2\lambda y = 0$, where λ is a constant

Let us assume

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} \quad \text{——— (2)}$$

as the solution of the given eqn (1)

Diff of (2) we get,

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting the value of $y, \frac{dy}{dx}, \frac{d^2y}{dx}$ in eqn (1)

$$\sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2} - 2x \cdot \sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1} + 2\lambda \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$\sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2} - 2\sum_{r=0}^{\infty} a_r(m+r)x^{m+r} + 2\lambda \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$\sum_{r=0}^{\infty} a_r\left[(m+r)(m+r-1)x^{m+r-2} - 2(m+r-\lambda)x^{m+r}\right] = 0 \quad (3)$$

Thus, Equating to zero the coefficient of lowest power of x i.e x^{m-2}

$$a_0(m)(m-1) = 0$$

Now $a_0 \neq 0$

$$\Rightarrow m(m-1) = 0$$

$$\boxed{\text{either } m=0 \text{ or } m=1}$$

$\therefore$ Equating to zero the coefficient of x^{m-1}

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$a_1(m+1)(m) = 0$

$\Rightarrow a_1 = 0$ or $m = 0$, but $m+1 \neq 0$

Now equating to zero the coeff of general term i.e x^{m+r}

$a_{r+2}[(m+r+2)(m+r+2-1)] - 2a_r(m+r-\lambda)$

$a_{r+2}(m+r+2)(m+r+1) - 2a_r(m+r-\lambda) = 0$

$$a_{r+2} = \frac{2(m+r-\lambda)a_r}{(m+r+2)(m+r+1)} \quad \text{--- (4)}$$

Case I :- when $m = 0$

$$a_{r+2} = \frac{2(r-\lambda)}{(r+2)(r+1)} \cdot a_r$$

Putting $r = 0, 2, 4, \ldots$

$$a_2 = \frac{-2\lambda a_0}{2\cdot 1} = -\frac{2\lambda}{2!}a_0$$

$$a_4 = \frac{2(2-\lambda)}{4\cdot 3}c_2 = \frac{(4-2\lambda)}{4\cdot 3}\left(\frac{-2\lambda}{2\cdot 1}\right)a_0$$

$$= \frac{-2(-2+\lambda)(-2\lambda)}{4\cdot 3\cdot 2\cdot 1}a_0$$

$$= \frac{(-2)^2(-2+\lambda)\lambda a_0}{4!}$$

$$= \frac{(-2)^2\lambda(\lambda-2)a_0}{4!}$$

$\lambda - 2 + 4$

$\vdots$

$$a_{2m} = \frac{(-2)^m \cdot \lambda(\lambda-2) \cdots (\lambda-2m+2)}{(2m)!} a_0$$

Again substituting $n = 1, 3, 5 \ldots$ in (4)

$$a_3 = \frac{2-2\lambda}{(1+2)(1+1)} a_1$$

$$= \frac{-2(\lambda-1)}{3!} a_1$$

$$a_5 = \frac{6-2\lambda}{5 \cdot 4} a_3 = \frac{(6-2\lambda)}{5 \cdot 4}\left(\frac{-2(\lambda-1)}{3!}\right) a_1$$

$$= \frac{-2(3-\lambda)(-2(\lambda-1))}{5!} a_1$$

$$= \frac{(-2)^2 (\lambda-1)(\lambda-3)}{5!} a_1$$

$\vdots$

$$\therefore \quad a_{2m+1} = \frac{(-2)^m (\lambda-1)(\lambda-3) \cdots (\lambda-2m+1)}{(2m+1)!} a_1$$

Now if $a_1 \neq 0$, we have

from (2)

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{2m} x^{2m} + \cdots$$
$$+ a_1 x + a_3 x^3 + a_5 x^5 + \cdots + a_{2m+1} x^{2m+1} + \cdots$$

$$y = a_0 + \left(\frac{-2\lambda}{2!} a_0\right) x^2 + \left(\frac{(-2)^2 \lambda(\lambda-2) a_0}{4!}\right) x^4$$
$$+ \cdots + \left(\frac{(-2)^m \lambda(\lambda-2) \cdots (\lambda-2m+2) a_0}{(2m)!}\right) x^{2m}$$
$$+ \cdots + a_1 x + \left(\frac{-2(\lambda-1) a_1}{3!}\right) x^3 + \left(\frac{(-2)^2(\lambda-1)(\lambda-3) a_1}{5!}\right) x^5$$
$$+ \cdots + \left(\frac{(-2)^m (\lambda-1)(\lambda-3) \cdots (\lambda-2m+1) a_1}{(2m+1)!}\right) x^{2m+1}$$
$$+ \cdots$$

$$y = a_0 \left[1 - \frac{2\lambda}{2!} x^2 + \frac{(2)^2 \lambda(\lambda-2)}{4!} x^4 + \cdots \right.$$
$$\left. \cdots + (-2)^m \lambda(\lambda-2) \cdots (\lambda-2m+2) x^{2m} + \cdots \right]$$
$$+ a_1 \left[x - \frac{2(\lambda-1)}{3!} x^3 + \frac{(2)^2 (\lambda-1)(\lambda-3)}{5!} x^5 \right.$$
$$\left. + \cdots + \frac{(-2)^m (\lambda-1)(\lambda-3) \cdots (\lambda-2m+1)}{(2m+1)!} x^{2m+1} + \cdots \right]$$

If $a_1 = 0$, then we have

$$y = a_0 \left[1 - \frac{2\lambda}{2!} x^2 + \frac{(2)^2 \lambda (\lambda - 2)}{4!} x^4 + \cdots \right.$$

$$\left. + \frac{(-2)^m \lambda (\lambda-2) \cdots (\lambda - 2m + 2)}{(2m)!} x^{2m} + \cdots \right]$$

$$y = y_1 \text{ (say)} \quad \text{———} \quad (*)$$

Case II :— When $m = 1$, from (4)

$$a_{r+2} = \frac{2(r+1) - 2\lambda}{(r+3)(r+2)} a_r$$

Putting $r = 1, 3, 5 \ldots$ we get

$$a_3 = \frac{2(2) - 2\lambda}{(1+3)(1+2)} a_1$$

$$= \frac{4 - 2\lambda}{(4)(3)} a_1 = 0 \quad \text{as } a_1 = 0$$

$$\therefore \ a_3 = a_5 = \cdots = 0.$$

Putting $r = 0, 2, 4, \ldots$ etc

$$a_2 = \frac{2 - 2\lambda}{3 \cdot 2} a_0 = \frac{-2(\lambda - 1)}{3!} a_0$$

$$a_4 = \frac{6 - 2\lambda a_2}{5 \cdot 4} = \frac{2^2 (\lambda - 1)(\lambda - 3)}{5!} a_0$$

$$a_{2m} = \frac{(-2)^m (\lambda-1)(\lambda-3) \cdots (\lambda-2m+1)\, a_0}{(2m+1)!}$$

we have

$$\left[y = \sum_{r=0}^{\infty} a_r x^{m+r} \quad \text{here } m=1 \quad y = \sum_{r=0}^{\infty} a_r x^{r+1} \right.$$

$$y = \sum_{r=0}^{\infty} a_r x^{r+1}$$

$$= a_0 x + a_2 x^3 + a_4 x^5 + \cdots + a_{2m} x^{2m+1} + \cdots$$

$$= a_0 \left\{ x - \frac{2(\lambda-1)}{3!} x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!} x^5 + \cdots + \frac{(-2)^m (\lambda-1)(\lambda-3) \cdots (\lambda-2m+1)}{(2m+1)!} x^{2m+1} + \cdots \right\}$$

$$= y_2 \quad \text{------} \quad (**)$$

Hence the general sol^n of Hermite equation is

$$y = Ay_1 + By_2$$

where A, B are arbitrary constant