Hahn Banach Theorem: If f is a continuous linear functional on a ~~normed line~~ subspace M of a normed linear space N, then f can be extended to a continuous linear functional g on N with $\|g\| = \|f\|$.

Special Case: Let M be a subspace of a normed linear space N, $x_0 \in N$ be s.t $x_0 \notin M$. and f be a continuous linear functional on M. Then $\exists$ a continuous linear functional g on $M + \langle x_0 \rangle$ s.t $g_{|M} = f$ and $\|g\| = \|f\|$

Proof:- Let $K = M + \langle x_0 \rangle$.

As $x_0 \notin M$

$\therefore$ $M + \langle x_0 \rangle$ is direct sum.

$\because$ If $m = \alpha x_0$ for some $0 \neq \alpha \in F$

Then $x_0 = \alpha^{-1} m \in M$.

which is not so.

$\therefore$ $M \cap \langle x_0 \rangle = \{0\}$

$\therefore$ $M + \langle x_0 \rangle$ is direct sum.

$\therefore$ each element of K can be uniquely written as $m + \alpha x_0$ for some $m \in M$ and $\alpha \in F$.

If $f: M \to F$ is s.t

$\|f\| = 0$

Then $f = 0$

Then taking $g: M + \langle x_0 \rangle \to F$ to be zero function,

we will have $g_{|M} = f = 0$, $\|g\| = \|f\| = 0$.

so we assume that $\|f\| \neq 0$

It is enough to prove the result when

$\|f\| = 1$.

$\because$ If $\|f\| = \beta \neq 0$ Then $\|\frac{f}{\beta}\| = 1$.

and $\frac{f}{\beta}: M \to F$ will be cont. linear

functional on M with norm 1,

therefore if result is proved for cont. functionals

of norm 1 on M, we will have a

cont. linear functional $g': M + \langle x_0 \rangle \to F$ with

$g'_{|M} = \frac{f}{\beta}$ and $\|g'\| = \|\frac{f}{\beta}\| = 1$.

Then $g = \beta g': M + \langle x_0 \rangle \to F$ will be a

cont. linear functional s.t

$g_{|M} = (\beta g')_{|M} = f$ and $\|g\| = \|\beta g'\|$

$= \beta . 1$

$= \beta = \|f\|$.

Note that if

$$g : K = M + \langle x_0 \rangle \longrightarrow F \text{ is defined as}$$

$$g(m + \alpha x_0) = f(m) + \alpha r \quad \text{for any } r \in F.$$

Then clearly g will be a linear functional on K which extends f.

We just need to choose r carefully, so that $\|g\| = 1$.

We just need to ensure that $\|g\| \leq 1$.

[$\because$ as g is ext. of f for any $r \in F$, $\|g\| \geq \|f\| = 1$.

i.e. we need to choose r carefully, so that

$$|g(m + \alpha x_0)| \leq \|m + \alpha x_0\| \quad \forall\, m \in M \;\&\; \alpha \in F. \qquad (1)$$

To start with assume that $F = \mathbb{R}$

Then (1) holds iff

$$-\|m + \alpha x_0\| \leq f(m) + \alpha r \leq \|m + \alpha x_0\| \quad \forall\, m \in M,\ \forall\, \alpha \in F.$$

$$\Rightarrow \quad -f(m) - \|m + \alpha x_0\| \leq \alpha r \leq -f(m) + \|m + \alpha x_0\|$$

If $\alpha > 0$.

Then $-f\left(\frac{m}{\alpha}\right) - \left\|\frac{m}{\alpha} + x_0\right\| \leq r \leq -f\left(\frac{m}{\alpha}\right) + \left\|\frac{m}{\alpha} + x_0\right\|$.

~~ie to~~ and if $\alpha < 0$, then

$$-f\left(\frac{m}{\alpha}\right) + \left\|\frac{m}{-\alpha} - x_0\right\| \geq r \geq -f\left(\frac{m}{\alpha}\right) - \left\|\frac{m}{-\alpha} - x_0\right\|.$$

$$-f\left(\frac{m}{\alpha}\right) + \left\|\frac{m}{\alpha} + x_0\right\| \geq r \geq -f\left(\frac{m}{\alpha}\right) - \left\|\frac{m}{\alpha} + x_0\right\|$$

ie $-f\left(\frac{m}{\alpha}\right) - \left\|\frac{m}{\alpha} + x_0\right\| \leq r \leq -f\left(\frac{m}{\alpha}\right) + \left\|\frac{m}{\alpha} + x_0\right\|$

Therefore to ensure that $\|g\| \leq 1$, we need to choose r s.t

$$-f\left(\frac{m}{\alpha}\right) - \left\|\frac{m}{\alpha} + x_0\right\| \leq r \leq -f\left(\frac{m}{\alpha}\right) + \left\|\frac{m}{\alpha} + x_0\right\|$$

$\forall\, m \in M$ & $\forall\, \alpha \in \mathbb{R}$. $\quad (*)$

Now, if $m_1, m_2 \in M$ are any elements,

then $f(m_2 - m_1) \leq |f(m_2 - m_1)|$

$\leq \|f\| \, \|m_2 - m_1\|$

$= \|m_2 - m_1\| \qquad [\because \|f\| = 1$

$$f(m_2) - f(m_1) \le \|m_2 - m_1\| = \|(m_2 + x_0) - (m_1 + x_0)\|$$

$$\le \|m_2 + x_0\| + \|m_1 + x_0\|.$$

$$\Rightarrow f(m_2) - f(m_1) \le \|m_2 + x_0\| + \|m_1 + x_0\|$$

$$\Rightarrow -f(m_1) - \|m_1 + x_0\| \le -f(m_2) + \|m_2 + x_0\|$$

(2) $\forall\ m_1, m_2 \in M$

If $a = \sup \{ -f(m) - \|m + x_0\| : m \in M \}$

& $b = \inf \{ -f(m) - \|m + x_0\| : m \in M \}$.

Then we have $a \le b$ [from (2)

If we choose $r \in \mathbb{R}$ s.t

$a \le r \le b$

Then clearly for any $m \in M$, $\alpha \in \mathbb{R}$ as $\frac{m}{\alpha} \in M$

$$\therefore -f\left(\frac{m}{\alpha}\right) - \left\|\frac{m}{\alpha} + x_0\right\| \le a \le r \le b \le -f\left(\frac{m}{\alpha}\right) - \left\|\frac{m}{\alpha} + x_0\right\|$$

$\forall\ m \in M$ & $\alpha \in \mathbb{R}$

ie If choose r s.t $a \le r \le b$

Then r satisfies $(*)$,

and hence as seen earlier, this would mean

$$|g(m+\alpha x_0)| \leq \|m+\alpha x_0\| \quad \forall\ m \in M$$
$$\& \ \alpha \in \mathbb{R}.$$

ie $|g(x)| \leq \|x\| \quad \forall\ x \in K.$

$\Rightarrow \quad \|g\| \leq 1.$

Also clearly $g_{|M} = f.$

$\Rightarrow \quad \|g\| \geq \|f\| = 1.$

$\Rightarrow \quad \|g\| = 1.$

Thus g will be cont. linear functional on K s.t. g is ext. of f on M & $\|g\| = \|f\| = 1.$

Now assume $F = \mathbb{C}$.