Riesz Representation Theorem

Let H be a Hilbert space and $f \in H^*$ be any element (ie f is continuous linear functional on H).

Then $\exists$ a unique $y \in H$ s.t

$$f(x) = \langle x, y \rangle \quad \forall \; x \in H$$

ie $f = f_y$ where $f_y : H \to F$ is defined as
$$f_y(x) = \langle x, y \rangle .$$

Proof:- First we show if such a 'y' exists, then it must be unique.

If $y, y' \in H$ are s.t.

$$f(x) = \langle x, y \rangle \quad \forall \; x \in H$$

and $f(x) = \langle x, y' \rangle \quad \forall \; x \in H$

Then $\langle x, y \rangle = \langle x, y' \rangle \quad \forall \; x \in H$

$$\Rightarrow \langle x, y - y' \rangle = 0 \quad \forall \; x \in H.$$

$$\Rightarrow \langle y - y', y - y' \rangle = 0$$

$$\Rightarrow y - y' = 0$$

$$\Rightarrow \boxed{y = y'}$$

If $f \underset{0}{\equiv} 0$

Then choose $y = 0$

As then $f(x) = 0 \ \forall \ x \in H$

and $\langle x, 0 \rangle = 0 \ \forall \ x \in H$.

$\Rightarrow \ f(x) = \langle x, 0 \rangle \ \forall \ x \in \underline{H}$.

<u>Assume</u> $f \neq 0$.

Then $\operatorname{Ker} f \neq H$ and $\operatorname{Ker} f$ is <u>a closed subspace</u> of <u>H</u>. (As f is continuous).

Let $N = \operatorname{Ker} f \neq H$.

Then $H = N \oplus N^{\perp}$ (As N is closed in H).

As $H \neq N$, $N^{\perp} \neq (0)$

Let $0 \neq y_0$ be a non-zero element in $N^{\perp}$.

As $\dfrac{H}{\operatorname{Ker} f} \cong \operatorname{Range} f$

and $\operatorname{Range} f$ is a subspace of F

and $\dim_F F = 1$ and $\operatorname{Range} f \neq 0$

$\Rightarrow \dim \operatorname{Range} f = 1$

$\Rightarrow \operatorname{Range} f = F$

$\Rightarrow \dim \left(\dfrac{H}{\operatorname{Ker} f} \right) = 1 \quad \Rightarrow \dim \left(\dfrac{H}{N} \right) = 1$

$\Rightarrow \dim (N^{\perp}) = 1$.

And $0 \neq y_0 \in N^\perp$

$\Rightarrow \quad N^\perp = \text{Span}\{y_0\}$

$\Rightarrow \quad H = N \oplus \text{span}\{y_0\}$

$H = \text{Ker} f \oplus \text{span}\{y_0\}$

<u>Claim</u> For a careful choice of α, $y = \alpha y_0$ is the required element.

For any choice of α,

$f(x) = 0 \quad \forall\, x \in \text{Ker} f$

and $\langle x, \alpha y_0 \rangle = \bar{\alpha} \langle x, y_0 \rangle = 0 \quad \forall\, x \in \text{Ker} f$

As $y_0 \in N^\perp = (\text{Ker} f)^\perp$

ie $f(x) = \langle x, \alpha y_0 \rangle \quad \forall\, x \in \text{Ker} f$.

for any choice of α.

$\therefore$ we need to make sure to find α in such a way that

$f(y_0) = \langle y_0, y \rangle$

$f(y_0) = \langle y_0, \alpha y_0 \rangle$

$\Rightarrow \quad f(y_0) = \bar{\alpha} \langle y_0, y_0 \rangle$

$\Rightarrow \quad \bar{\alpha} = \dfrac{f(y_0)}{\|y_0\|^2} \quad \Rightarrow \quad \alpha = \dfrac{\overline{f(y_0)}}{\|y_0\|^2}$.

we show for this choice of α,

$$y = \alpha y_0 = \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0 \quad \text{is s.t.}$$

$$f(x) = \langle x, y \rangle \quad \forall \; x \in H.$$

~~Pf.~~ If $x \in H$ is any element,

then as $H = N \oplus \text{span}\{y_0\}$

$\therefore \quad x = m + \beta y_0$ for some $m \in N$ and $\beta \in F$

$$\langle x, \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0 \rangle = \langle m + \beta y_0, \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0 \rangle$$

$$= \langle m, \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0 \rangle + \langle \beta y_0, \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0 \rangle$$

$$= \frac{f(y_0)}{\|y_0\|^2} \langle m, y_0 \rangle + \beta \frac{f(y_0)}{\|y_0\|^2} \langle y_0, y_0 \rangle.$$

$$= 0 + \beta f(y_0) \qquad \left[\begin{array}{l} \langle m, y_0 \rangle = 0 \\ m \in N \\ \text{and } y_0 \in N^{\perp}. \end{array} \right.$$

$$= f(\beta y_0)$$

$$= f(m + \beta y_0) \qquad (m \in \text{Ker} f)$$

$$= f(x)$$

$$\Rightarrow f(x) = \langle x, \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0 \rangle \quad \forall \; x \in H$$

$$\therefore \boxed{y = \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0}$$