Section B

Measurable Functions

Let f be the extended real valued function whose domain is E, a measurable set i.e $f: E \to \mathbb{R} \cup \{-\infty, \infty\}$ then a function f is said to be measurable function if the set $\{x \in E : f(x) > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$.
i.e $f^{-1}(\alpha, \infty]$ is measurable $\forall \alpha \in \mathbb{R}$

Proposition 1: Let f be a extended real value f_x n whose domain is E, a measurable set, then the following statements are equivalent

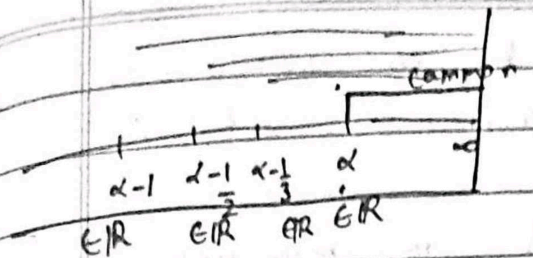
1. $\forall \alpha \in \mathbb{R}$, the set $\{x \in E, f(x) > \alpha\}$ is measurable
2. $\forall \alpha \in \mathbb{R}$, the set $\{x \in E, f(x) \geq \alpha\}$ is measurable
3. $\forall \alpha \in \mathbb{R}$, the set $\{x \in E, f(x) < \alpha\}$ is measurable
4. $\forall \alpha \in \mathbb{R}$, the set $\{x \in E, f(x) \leq \alpha\}$ is measurable

Pf. :- ① $\Rightarrow$ ②

Pf: Given that the set $x \in E, f(x) > \alpha$ is measurable $\forall \alpha \in \mathbb{R}$

To prove: $\{x \in E, f(x) \geq \alpha\}$ is measurable

We know $\{x \in E, f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E, f(x) > \alpha - \frac{1}{n}$



By hypothesis, each set on the right hand side is measurable and intersection of measureable set is measurable sets is measurable

$\therefore$ The set on the R.H.S is measurable

Hence the set $\{x \in E, f(x) \geq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

(2) $\Rightarrow$ (3)

Given The set $\{x \in E, f(x) \geq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

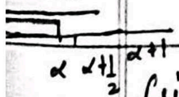
Now $\{x \in E, f(x) < \alpha\} = \{x \in E, f(x) \geq \alpha\}^c$

Since by hypothesis $\{x \in E, f(x) \geq \alpha\}$ is measurable and the complement of a measurable set is measurable

$\therefore$ The set $\{x \in E, f(x) \geq \alpha\}^c$ is measurable

Hence the set $\{x \in E, f(x) < \alpha\}$ is measurable

(3) $\Rightarrow$ (4)



Given that the set $\{x \in E, f(x) < \alpha\}$ is measurable

Now $\{x \in E, f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E, f(x) < \alpha + \frac{1}{n}\}$

Each set on the R.H.S is measurable and intersection of a measurable set is measurable

$\therefore$ The set on the R.H.S is measurable

$\Rightarrow$ The set on the L.H.S is also measurable

$\therefore$ The set $\{x \in E, f(x) \leq \alpha\}$ is also measurable

④ $\Rightarrow$ ①

Given that the set $\{x \in E, f(x) \leq \alpha\}$ is measurable

$\{x \in E, f(x) > \alpha\} = \{x \in E, f(x) \leq \alpha\}^c$

Since by hypothesis $\{x \in E, f(x) \leq \alpha\}$ is measurable and the complement of a measurable set is measurable

$\therefore$ The

RK (i) In the view of above proposition, any one of the above four statements can be taken as def. of measurable f_x^{ns}

(ii) ① $f^{-1}(\alpha, \infty]$
② $f^{-1}[\alpha, \infty]$
③ $f^{-1}[-\infty, \alpha)$
④ $f^{-1}[-\infty, \alpha]$

These sets are equivalent to ①, ②, ③, ④