Cayley - Hamilton Theorem :-

Statement :- Every square matrix satisfies its characteristic equation.

Proof. Let A be any square matrix of order n, and its characteristic equation be

$$p_0 + p_1\lambda + p_2\lambda^2 + \cdots + p_n\lambda^n = 0$$

We have to prove that A satisfies this equation

i.e, $$p_0 I + p_1 A + p_2 A^2 + \cdots + p_n A^n = 0 \quad \text{--- (1)}$$

For proving this, we proceed as follow:

We know that $(A-\lambda I)\,\text{adj}.(A-\lambda I) = |A-\lambda I|\,I$

$[\because A\,\text{adj}.A = |A|I]$

Let $\text{adj}.(A-\lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}$

$\therefore$ We have, $(A-\lambda I)(B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1})$

$$= (p_0 + p_1\lambda + p_2\lambda^2 + \cdots + p_n\lambda^n)I$$

$AB_0 + AB_1\lambda + AB_2\lambda^2 + \cdots + AB_{n-1}\lambda^{n-1} - \lambda B_0 - B_1\lambda^2$
$- B_2\lambda^3 + \cdots + B_{n-2}\lambda^{n-1} - B_{n-1}\lambda^n$

$$= (p_0 + p_1\lambda + p_2\lambda^2 + \cdots + p_{n-1}\lambda^{n-1} + p_n\lambda^n)I$$

$$AB_0 + (AB_1 - B_0)\lambda + (AB_2 - B_1)\lambda^2 + \cdots \cdots + (AB_{n-1} - B_{n-2})\lambda^{n-1} - B_{n-1}\lambda^n$$

$$= (p_0 + p_1\lambda + p_2\lambda^2 + \cdots + p_{n-1}\lambda^{n-1} + p_n\lambda^n)I$$

Equating Coeff. of like powers of λ, we get

$$AB_0 = p_0 I$$
$$AB - B_0 = p_1 I$$
$$AB_2 - B_1 = p_2 I$$
$$\vdots$$
$$AB_{n-1} - B_{n-2} = p_{n-1} I$$
$$-B_{n-1} = p_n I$$

Pre-multiplying above equations by $I, A, A^2, \cdots, A^n$ respectively

$$AB_0 = p_0 I$$
$$A^2B_1 - AB_0 = Ap_1$$
$$A^3B_2 - A^2B_1 = A^2 p_2$$
$$\vdots$$
$$A^n B_{n-1} - A^{n-1} B_{n-2} = A^{n-1} p_{n-1}$$
$$-A^n B_{n-1} = A^n p_n$$

Now, Adding these equations, we get

$$\cancel{A}\cancel{B_0} + \cancel{A^2}B_1 - \cancel{A}\cancel{B_0} + \cancel{A^3}B_2 - \cancel{A^2}\cancel{B_1} + \cdots + \cancel{A^n}B_{n-1} - \cancel{A^{n-1}}B_{n-2} - A^n \cancel{B_{n-1}}$$

$$= p_0 I + p_1 A + p_2 A^2 + p_n A^n.$$

$$\Rightarrow \quad p_0 I + p_1 A + p_2 A^2 + \cdots + p_n A^n$$

which is same as (1)

Hence the theorem.