Chapter - 13

Bessel's Equation & Bessel's functions :-

* Bessel's Equation :-

The differential equation of the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \quad \text{OR}$$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2) y = 0$$ is known

Bessel's differential equation.

Solution of this differential equation are called Bessel's function of order n and one solution is denoted by $J_n(x)$ which is known as Bessel's function of the first kind and another solution is denoted by $J_{-n}(x)$, which is known as Bessel's function of the second kind.

Solution of Bessel's Differential Equation :-

The Bessel's Diff eq^n is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \text{—— (1)}$$

Let the sol^n of eq^n (1) is given by

$$y = \sum_{k=0}^{\infty} a_k x^{k+m} \quad \text{—— (*)}$$

$$\therefore y' = \sum_{k=0}^{\infty} a_k x^{k+m-1}(k+m)$$

$$y'' = \sum_{k=0}^{\infty} a_k (k+m)(k+m-1) x^{k+m-2} \quad \text{—— (2)}$$

Put (2) in (1)

$$x^2\left[\sum_{k=0}^{\infty} a_k (k+m)(k+m-1) x^{k+m-2}\right] + x\left[\sum_{k=0}^{\infty} a_k \cancel{x}(k+m)\, x^{k+m-1}\right] + (x^2 - n^2)\left(\sum_{k=0}^{\infty} a_k x^{k+m}\right) = 0$$

$$\sum_{k=0}^{\infty} a_k (k+m)(k+m-1) x^{k+m} + \sum_{k=0}^{\infty} a_k (k+m) x^{k+m} + \sum_{k=0}^{\infty} a_k x^{k+m+2} - n^2 \sum_{k=0}^{\infty} a_k x^{k+m} = 0$$

$$\sum_{k=0}^{\infty} a_k \left[(k+m)(k+m-1)+(k+m)-n^2\right]x^{k+m} + \sum_{k=0}^{\infty} a_k x^{k+m+2} = 0$$

$$\sum_{k=0}^{\infty} a_k \left[(k+m)(k+m-\cancel{1}+\cancel{1})-n^2\right]x^{k+m} + \sum_{k=0}^{\infty} a_k x^{k+m+2} = 0$$

$$\sum_{k=0}^{\infty} a_k \left[(k+m)(k+m)-n^2\right]x^{k+m} + \sum_{k=0}^{\infty} a_k x^{k+m+2} = 0$$

$$\sum_{k=0}^{\infty} a_k \left((k+m)^2-n^2\right)x^{k+m} + \sum_{k=0}^{\infty} a_k x^{k+m+2} = 0 \qquad (3)$$

Comparing the coeff of x^m

$k = \cancel{k+2-m}$

$a_0((0+m)^2-n^2)=0$ $\quad (k\to 0)$

$a_0(m^2-n^2)=0$

but $a_0 \neq 0$

$\Rightarrow m^2-n^2=0$

$\Rightarrow m^2=n^2$

$m=\pm n$

i.e $\boxed{m=n}$ or $\boxed{m=-n}$

Comparing the coeff of x^{m+1} $(k\to 1)$

$a_1((m+1)^2-n^2)=0$

$a_1(m^2+2m+1-n^2)=0$

$m^2+2m+1-n^2 \neq 0$

$\Rightarrow \boxed{a_1=0}$

if $m=n$

$\cancel{n^2}+2n+1-\cancel{n^2}=0$

$2n+1\neq 0$

if $m=-n$

$\cancel{n^2}-2n+1-\cancel{n^2}=0$

$-2n+1\neq 0$

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Comparing Coeff. of x^{k+m+2} $(k \to k+2)$

$$a_{k+2}\left((k+2+m)^2-n^2\right)+a_k=0$$

$$a_{k+2}=\frac{-1}{(k+2+m)^2-n^2}\,a_k \quad \text{——} (4)$$

Put $k=1,3,5,7\ldots$ in (4), we get

$$a_3=\frac{-1}{(1+2+m)^2-n^2}\,a_1$$

$$a_3=0 \quad [\because a_1=0]$$

$$a_5=\frac{-1}{(3+2+m)^2-n^2}\,a_3=0 \quad [\text{as } a_3=0]$$

IIly $a_7=a_9=\cdots=0$ —— (5)

<u>Now two Cases arise :-</u>

<u>Case I</u> :- Put $m=n$ in eqn (4)

$$a_{k+2}=\frac{-1}{(k+2+n)^2-n^2}\,a_k$$

Now Put $k=0,2,4,6,\ldots$

$$a_2=\frac{-1}{(0+2+n)^2-n^2}\,a_0$$

$$-\frac{1}{(2+n)^2-n^2}a_0 = -\frac{1}{4+\cancel{n^2}+4n-\cancel{n^2}}a_0$$

$$a_2 = -\frac{1}{4(n+1)}a_0$$

$$a_4 = -\frac{1}{(2+2+n)^2-n^2}a_2$$

$$= -\frac{1}{(4+n)^2-n^2}a_2 = -\frac{1}{16+\cancel{n^2}+8n-\cancel{n^2}}a_2$$

$$= \left(-\frac{1}{16+8n}\right)\left(-\frac{1}{4(n+1)}\right)a_0$$

$$= \left(\frac{1}{8(n+2)}\right)\left(\frac{1}{4(n+1)}\right)a_0$$

$$= \frac{1}{4\cdot 8(n+1)(n+2)}a_0 \quad \text{—— (6)}$$

and so on

Now from eqn (*) and as $m=n$

$$y = \sum_{k=0}^{\infty} a_k x^{n+k}$$

$$= a_0x^n + a_1x^{n+1} + a_2x^{n+2} + \cdots$$

$$= a_0x^n + 0 + \left(-\frac{1}{4(n+1)}\right)a_0x^{n+2} + \frac{1}{4\cdot 8(n+1)(n+2)}a_0x^{n+4} + \cdots$$

$$= a_0 \left[x^n - \frac{1}{4(n+1)} x^{n+2} + \frac{1}{4 \cdot 8 (n+1)(n+2)} x^{n+4} + \cdots \right] \quad (7)$$

Put $a_0 = \frac{1}{2^n \lceil n+1}$ in eq^n (7)

$$J_n(x) = y = \frac{1}{2^n \lceil n+1} x^n \left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \frac{1}{2(n+1)(n+2)} \left(\frac{x}{2}\right)^4 - \cdots \right]$$

$$\boxed{J_n(x) = y = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \lceil n+r+1} \left(\frac{x}{2}\right)^{2r+n}}$$

Case 2 :- when $m = -n$

changing n to $-n$ in eq^n (7)

$$y = a_0 \left[x^{-n} - \frac{1}{4(-n+1)} x^{-n+2} + \frac{1}{4 \cdot 8 (-n+1)(-n+2)} x^{-n+4} + \cdots \right] \quad (8)$$

Put $a_0 = \frac{1}{2^{-n} \lceil (-n+1)}$ in eq^n (8)

$$y = \frac{1}{2^{-n} \lceil (-n+1)} x^{-n} \left[1 - \frac{1}{(-n+1)} \left(\frac{x}{2}\right)^2 + \frac{1}{(-n+1)(-n+2)} \left(\frac{x}{2}\right)^4 - \cdots \right]$$

$$J_{-n}(x) = y = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

Result :- $\boxed{\Gamma(n+1) = n\,\Gamma(n)}$

Recurrence Relations :-

(1) $x J_n'(x) = n.J_n(x) - x J_{n+1}(x)$

proof: we know that

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

where n is a +ve Integer

Diff w.r.t 'x', we have

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!\,\Gamma(n+r+1)} \cdot \frac{1}{2} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \quad (*)$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} (-1)^r \frac{2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$(2)^{n+2r\cancel{-1+1}}$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n)}{r! \,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^r \, 2r}{r! \,\Gamma(n+r+1)} \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n \sum_{r=0}^{\infty} \frac{(-1)^r \, 1}{r! \,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$+ x \sum_{r=1}^{\infty} \frac{(-1)^r \, 2r}{r! \,\Gamma(n+r+1)} \left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n(x) + x \sum_{r=1}^{\infty} (-1)^r \cdot \frac{\cancel{2}\,\cancel{r}}{\cancel{r}\,(r-1)! \,\Gamma(n+r+1)} \left(\frac{1}{\cancel{2}}\right)\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n(x) + x \sum_{r=1}^{\infty} (-1)^r \frac{1}{(r-1)! \,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

Put $r-1 = s$

$r = s+1$

$$= n J_n(x) + x \sum_{s=0}^{\infty} (-1)^{s+1} \frac{1}{s! \,\Gamma(n+s+1+1)} \left(\frac{x}{2}\right)^{n+2(s+1)-1}$$

$$= n J_n(x) + x \sum_{s=0}^{\infty} (-1)^s (-1)^1 \frac{1}{s! \,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+2s+2-1}$$

$$= n J_n(x) - x \sum_{s=0}^{\infty} (-1)^s \frac{1}{s! \,\Gamma((n+1)+s+1)} \left(\frac{x}{2}\right)^{n+2s+1}$$

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$$= n J_n(x) - x J_{n+1}(x)$$

$$\Rightarrow x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

(2) $$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

As in I, we have

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

(see eq (*) in R.R-I)

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(2n+2r-n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$+ \sum_{r=0}^{\infty} (-1)^r \frac{(2n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n J_n(x) + \sum_{r=0}^{\infty} (-1)^r \frac{\cancel{2}(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \left(\frac{x}{\cancel{2}}\right)$$

$$= -n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\,\underline{\lfloor n+r}} \left(\frac{x}{2}\right)^{n+2r-1} (x)$$

$$\left[\because \Gamma(n+r+1) = \underline{\lfloor n+r}\right]$$

$$= -n J_n(x) + x \sum_{r=0}^{\infty} (-1)^r \frac{\cancel{(n+r)}}{r!\,\cancel{(n+r)}\,\underline{\lfloor n+r-1}} \left(\frac{x}{2}\right)^{n+2r-1}$$

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$$= -n\,J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,|\underline{n+r-1}} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$\left[\because \Gamma(n+r) = |\underline{n+r-1}\right]$$

$$= -n\,J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -n\,J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n-1+2r}$$

$$\boxed{x J'_n(x) = -n\,J_n(x) + x\,J_{n-1}(x)}$$

$$\left[\because J_{n-1}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n-1+2r}\right]$$

(3) $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

from Recurrence relation I, we have

$$x J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x) \quad \text{(1)}$$

and from R.R – II, we have

$$x J'_n(x) = -n\,J_n(x) + x\,J_{n-1}(x) \quad \text{(2)}$$

Add (1) and (2)

$$x J'_n(x) + x J'_n(x) = \cancel{n\,J_n(x)} - \cancel{n\,J_n(x)} - x\,J_{n+1}(x) + x\,J_{n-1}(x)$$

$$2x J_n'(x) = -x J_{n+1}(x) + x J_{n-1}(x)$$
$$2x J_n'(x) = x J_{n-1}(x) - x J_{n+1}(x)$$
$$\boxed{2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)}$$

(4) $$2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

from R.R-I and R.R-II, we have

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{--- (1)}$$
$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \quad \text{--- (2)}$$

Sub (1) and (2)

$$x J_n'(x) - x J_n'(x) = n J_n(x) + n J_n(x) - x J_{n+1}(x) - x J_{n-1}(x)$$

$$0 = 2n J_n(x) - x[J_{n+1}(x) + J_{n-1}(x)]$$

$$x[J_{n+1}(x) + J_{n-1}(x)] = 2n J_n(x)$$

$$\boxed{2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]}$$

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(3) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

proof L.H.S $\frac{d}{dx}\left[x^{-n} J_n(x)\right]$

$= J_n(x)\frac{d}{dx}x^{-n} + x^{-n}\frac{d}{dx}J_n(x)$

$= J_n(x)(-n x^{-n-1}) + x^{-n} J_n'(x)$

$= -n x^{-n-1} J_n(x) + x^{-n} J_n'(x)$

$= x^{-n-1}\left[-n J_n(x) + x J_n'(x)\right]$

$= x^{-n-1}\left[-n J_n(x) + x J_n'(x)\right]$

$= x^{-n-1}\left[-n J_n(x) + (n J_n(x) - x J_{n+1}(x))\right]$

[from Rec. Relation I, we have $x J_n'(x) = n \cdot J_n(x) - x J_{n+1}(x)$]

$= x^{-n-1}\left[-n \cancel{J_n(x)} + n \cancel{J_n(x)} - x J_{n+1}(x)\right]$

$= x^{-n-1}\left[-x J_{n+1}(x)\right]$

$= (-x)(x^{-n-1}) J_{n+1}(x)$

$= -x^{-n-1+1} J_{n+1}(x)$

$= -x^{-n} J_{n+1}(x)$

$$\boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)}$$

6) $$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\frac{d}{dx}[x^n J_n(x)] = n x^{n-1} J_n(x) + x^n J_n'(x)$$

$$= x^{n-1}[n J_n(x) + x J_n'(x)]$$

$$= x^{n-1}[n J_n(x) + \{-n J_n(x) + x J_{n-1}(x)$$

$\Big\{$ from R.R. II, we have

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \Big\}$$

$$= x^{n-1}[\cancel{n J_n(x)} - \cancel{n J_n(x)} + x J_{n-1}(x)]$$

$$= x^{n-1}[x J_{n-1}(x)]$$

$$= x^n J_{n-1}(x)$$

$$\boxed{\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)}$$

Generating function for $J_n(x)$:-

The function $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ is called generating function i.e coefficient of z^n is $J_n(x)$ and coefficient of z^{-n} is $(-1)^n J_n(x)$, where n is positive integer.

proof :- $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{xz}{2}-\frac{x}{2z}}$

$$= e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}}$$

$$= \left[1+\frac{xz}{2}+\frac{1}{2!}\left(\frac{xz}{2}\right)^2+\dots+\frac{1}{n!}\left(\frac{xz}{2}\right)^n+\frac{1}{(n+1)!}\left(\frac{xz}{2}\right)^{n+1} + \frac{1}{(n+2)!}\left(\frac{xz}{2}\right)^{n+2}+\dots\right] \times$$

$$\left[1-\frac{x}{2z}+\frac{1}{2!}\left(\frac{x}{2z}\right)^2+\dots+\frac{(-1)^n}{n!}\left(\frac{x}{2z}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2z}\right)^{n+1}+\frac{(-1)^{n+2}}{(n+2)!}\left(\frac{x}{2z}\right)^{n+2}+\dots\right]$$

$$\left\{ \begin{array}{l} \text{Because we know that} \\ e^x = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots \\ \text{and } e^{-x} = 1-x+\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}+\dots \end{array} \right\}$$

Coefficient of z^n in this product

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+1} + \frac{1}{(n+2)!}\frac{1}{2!}\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2} + \cdots$$

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \frac{(-1)^0}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{\Gamma n+2}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma n+3}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$\left\{ \begin{array}{l} \Gamma n+1 = n! \\ \Gamma n+2 = n+1! \\ \Gamma n+3 = n+2! \end{array} \right\}$$

$$= \frac{(-1)^0}{0!\,\Gamma n+1}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,\Gamma n+2}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma n+3}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \frac{(-1)^0}{0!\,\Gamma n+0+1}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,\Gamma n+1+1}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma n+2+1}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma n+r+1}\left(\frac{x}{2}\right)^{n+2r}$$

$$= J_n(x)$$

$$\left\{ \because J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \right\}$$

Similarly, the coefficient of z^{-n} in this product

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)\cdot\left(\frac{x}{2}\right)^{n+1} + \frac{(-1)^{n+2}}{(n+2)!}\cdot\frac{1}{2!}\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2} + \cdots$$

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{(n+2)!}\cdot\frac{1}{2!}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= (-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{(n+2)!}\cdot\frac{1}{2!}\left(\frac{x}{2}\right)^{n+4} + \cdots\right.$$

$$= (-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{\Gamma n+1}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma n+3}\left(\frac{x}{2}\right)^{n+4} + \cdots\right]$$

$$= (-1)^n\left[\frac{1}{0!\,\Gamma n+1}\left(\frac{x}{2}\right)^n + \frac{(-1)}{1!\,\Gamma(n+1)}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma n+3}\left(\frac{x}{2}\right)^{n+4} + \cdots\right]$$

$= (-1)^n J_n(x)$