V & W are V.S over the same field F

A map $T: V \longrightarrow W$ is called a linear transformation

if $T(x+y) = T(x) + T(y) \ \forall \ x, y \in V$

& $T(\alpha x) = \alpha T(x) \ \forall \ \alpha \in F, \forall x \in V$

$(N, \|\cdot\|)$ & $(N', \|\cdot\|')$ are two normed linear spaces.

$T: N \longrightarrow N'$ is continuous.

if for any seq. $\{x_n\}$ converging to x in N implies $\{T(x_n)\}$ converges to $T(x)$ in N'.

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A map $T: N \longrightarrow N'$, where N and N' are normed linear spaces, is called bounded if

$\exists \ K > 0$ s.t.

$$\|T(x)\| \leq K \cdot \|x\| \quad \forall \ x \in N$$

Example: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as

$T(x, y) = 2x + 3y$.

Let $u \in \mathbb{R}^2$ be any element

$u = (x, y)$

$$\|T(u)\| = \|2x + 3y\|$$

$$= |2x + 3y|$$

$$\leq 2|x| + 3|y| \leq \sqrt{2^2 + 3^2} \sqrt{x^2 + y^2}$$

[Cauchy-Schwarz inequality]

$\|T(u)\| \leq \sqrt{13}\ \|(x,y)\| = \sqrt{13}\ \|u\|$

$\Rightarrow \|T(u)\| \leq \sqrt{13}\ \|u\| \quad \forall\ u \in \mathbb{R}^2.$

$\Rightarrow$ T is ~~cont~~ Bounded function.

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Thm:- Let $(N, \|\ \|)$ and $(N', \|\ \|')$ be two normed linear spaces. $T: N \rightarrow N'$ be a linear transformation. Then the following are equivalent:

(i) T is continuous.

(ii) T is continuous at origin ie 0.

(iii) T is bounded.

(iv) $T(S)$ is a bounded subset of N' where $S = \{x \in N : \|x\| \leq 1\}$. [$S = B[0,1]$ in N]

(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Let $\{x_n\}$ be a seq. in N s.t

$\{x_n\} \longrightarrow x.$

$\Rightarrow \|x_n - x\| \longrightarrow 0$ as $n \rightarrow \infty$.

$\Rightarrow \{x_n - x\} \longrightarrow 0.$

But T is continuous at 0.

$\Rightarrow \{T(x_n - x)\} \longrightarrow T(0) = 0$

$\Rightarrow \{T(x_n) - T(x)\} \longrightarrow 0$ [T is a L.T].

$\Rightarrow \{T(x_n) - T(x) + T(x)\} \longrightarrow 0 + T(x) = T(x)$

$\Rightarrow \{T(x_n)\} \longrightarrow T(x)$

Thus $\{x_n\} \to x \Rightarrow \{T(x_n)\} \to T(x)$

$\Rightarrow$ T is continous.

(ii) $\Rightarrow$ (iii) If T is not bounded.

Then for any $n \in \mathbb{N}$,

$\exists \; x_n \in N$ s.t

$\|T(x_n)\| > n \|x_n\|$.

$$\Rightarrow \frac{\|T(x_n)\|}{n \|x_n\|} > 1.$$

$$\Rightarrow \left\| \frac{T(x_n)}{n \|x_n\|} \right\| > 1.$$

$$\Rightarrow \left\| T\left(\frac{x_n}{n \|x_n\|}\right) \right\| > 1 \quad \forall \, n \in \mathbb{N}$$

Let $y_n = \dfrac{x_n}{n \|x_n\|}$, then $\|T(y_n)\| > 1$ $\forall \, n \in \mathbb{N}$ — (*)

Also $y_n = \dfrac{x_n}{n\,\|x_n\|}$

$$\|y_n\| = \frac{1}{n\,\|x_n\|} \cdot \|x_n\|$$

$$= \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow \{y_n\} \longrightarrow 0$

But $(*) \Rightarrow \|T(y_n)\| > 1 \quad \forall\, n \in \mathbb{N}$.

~~But~~ $\Rightarrow \{T(y_n)\} \not\longrightarrow 0$

Which is a contradiction to the hypothesis that T is continuous at origin.

$\therefore$ T must be bounded.

(iii) $\Rightarrow$ (ii) Assume that T is bounded,

$\Rightarrow \exists\; K > 0$ s.t

$$\|T(x)\| \leq K\,\|x\| \quad \forall\, x \in N. \quad (1)$$

Let $\{x_n\}$ be a seq. in N s.t

$$\{x_n\} \longrightarrow 0.$$

$(1) \Rightarrow \|T(x_n)\| \leq K\,\|x_n\| \quad \forall\, n \in \mathbb{N}.$

$$0 \leq \|T(x_n)\| \leq K\,\|x_n\| \longrightarrow 0 \quad \text{as } n \to \infty.$$

By Squeeze principle, $\|T(x_n)\| \longrightarrow 0$ as $n \to \infty$

$\{T(x_n)\} \longrightarrow 0 = T(0).$

Thus $\{x_n\} \longrightarrow 0 \Rightarrow \{T(x_n)\} \longrightarrow T(0).$

$\Rightarrow$ T is continuous at 0.

(iii) $\Rightarrow$ (iv) Given T is bounded,

$\exists\ K > 0$ s.t

$\|T(x)\| \leq K\|x\| \quad \forall\ x \in N.$

Let $y \in T(S)$ be any element,

$y = T(x_0)$ for some $x_0 \in B[0,1]$.

$\|y\| = \|T(x_0)\|.$

$\leq K\|x_0\|$

But $x_0 \in B[0,1]$

$\Rightarrow \|x_0\| \leq 1.$

$\Rightarrow \|y\| \leq K \cdot 1$

$\Rightarrow \|y\| \leq K \quad \forall\ y \in T(S)$

$\Rightarrow T(S)$ is bounded.

(iv) $\Rightarrow$ (iii) Given $T(S)$ is bounded.

$\therefore \exists\ K > 0$ s.t

$$\|y\| \leq K \quad \forall \; y \in T(S)$$

$$\Rightarrow \|T(x)\| \leq K \quad \forall \; x \in S \qquad (2)$$

let $0 \neq x' \in N$ be any element.

Then $x_0 = \dfrac{x'}{\|x'\|}$ is s.t

$$\|x_0\| = 1 \Rightarrow x_0 \in S$$

$$(2) \Rightarrow \|T(x_0)\| \leq K.$$

$$\Rightarrow \left\| T\left(\frac{x'}{\|x'\|}\right)\right\| \leq K.$$

$$\Rightarrow \left\| \frac{1}{\|x'\|} \cdot T(x')\right\| \leq K.$$

$$\Rightarrow \frac{\|T(x')\|}{\|x'\|} \leq K.$$

$$\Rightarrow \|T(x')\| \leq K \|x'\| \quad \forall \; 0 \neq x' \in N$$

$\Rightarrow$ Also $\|T(0)\| = \|0\| = K \cdot \|0\|$

$$\Rightarrow \|T(x)\| \leq K \|x\| \quad \forall \; x \in N$$

$\Rightarrow$ T is bounded

(iv) $\Rightarrow$ (i) Given $T(S)$ is bounded.

$\exists$ $K > 0$ s.t

$\|T(x)\| \leq K \quad \forall \; x \in S = B[0,1]$ —— 3

let $\{x_n\}$ be a sequence in N s.t

$\{x_n\} \longrightarrow x.$

$\Rightarrow \quad \|x_n - x\| \longrightarrow 0 \quad$ as $n \to \infty$.

For any $\epsilon > 0$, $\exists$ $k \in \mathbb{N}$ s.t.

$\|x_n - x\| < \frac{\epsilon}{K} \quad \forall \; n \geq k.$

$\Rightarrow \left\| \frac{K}{\epsilon}(x_n - x) \right\| < 1 \quad \forall \; n \geq k.$

$\frac{K}{\epsilon}(x_n - x) \in B[0,1] \quad \forall \; n \geq k.$

3 $\Rightarrow \left\| T\left(\frac{K}{\epsilon}(x_n - x)\right) \right\| \leq K \quad \forall \; n \geq k.$

$\Rightarrow \left\| \frac{K}{\epsilon} T(x_n - x) \right\| \leq K \quad \forall \; n \geq k.$

$\Rightarrow \frac{K}{\epsilon} \|T(x_n - x)\| \leq K \quad \forall \; n \geq k.$

$\Rightarrow \|T(x_n) - T(x)\| \leq \epsilon \quad \forall \; n \geq k.$

$\Rightarrow \{T(x_n)\} \to T(x) \Rightarrow$ T is continuous