exam # Equipment Renewal Problem:

Thm: The renewing rate of a machine is asymptotically reciprocal of the mean life of the machinery.

Proof. Let the pdf of failure time of a machine be $f(x)$ i.e $x_1, x_2, \ldots$ be the life span of the machines then each machine has $f(x)$ as its pdf.

Let the machine fails $n-1$ times during the interval $(0, t)$ and at each failure an immediate replacement is being made with similar machine.

Thus $x_1 + x_2 + \cdots + x_{n-1} < t$

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& $x_1 + x_2 + \cdots + x_{n-1} + x_n > t$

The prob. that a renewal will occur during the interval $(t, t+\delta t)$ will be called replacement rate and we denote it by $h(t)\,dt$ where $h(t)$ is known as renewal density function. If we write

$$P\left[t \le \sum_{i=1}^{r} x_i \le t + dt\right] = f_r(t)\,dt \text{ then}$$

$$h(t)\,dt = \left[\sum_{r=1}^{\infty} f_r(t)\right] dt$$

or $h.t = \sum_{r=1}^{\infty} f_r(t)$

As t becomes very large, the renewal density function $h(t) \to$ constant. To prove this result, we derive the Laplace - Stieltjes Transformations of $h(t)$ & $f(t)$ given by

$$h^*(z) = \int_0^{\infty} e^{-zt}\, h(t)\,dt$$

& $f^*(z) = \int_0^{\infty} e^{-zt} f(t)\,dt$ and also by

the property of this transformation, we have $f_r^*(z) = \int_0^{\infty} e^{-zt} f_r(t)\,dt = \{f^*(z)\}^r$

$$\therefore h^*(z) = \int_0^{\infty} e^{-zt}\left[\sum_{r=1}^{\infty} f_r(t)\right] dt$$

$$= \sum_{r=1}^{\infty} \{f^*(z)\}^r = \frac{f^*(z)}{1 - f^*(z)}$$ (infinite GP)

Let the mean life of the machine be given by $\lambda = \int_0^\infty x f(x)\,dx$

Now $\lim_{z \to 0} f^*(z) = \lim_{z \to 0} \int_0^\infty e^{-zt} f(t)\,dt$

$$= \int_0^\infty f(t)\,dt$$

$\therefore$ As $z \to 0$, $h^*(z) = \dfrac{\int_0^\infty f(t)\,dt}{1 - \int_0^\infty f(t)\,dt}$

Now since $f(t)$ is the pdf of failure time and we know that the time between successive arrivals follow exponential dist. $\therefore$ we have

$\int_0^\infty f(t)\,dt \to e^{-\lambda z}$ (asymptotically)

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & ; 0 \le t < \infty \\ 0 & , \text{otherwise} \end{cases}$$

$\therefore$ As $z \to 0$,

$\cancel{f^*(z}$ $h^*(z) \to \dfrac{e^{-\lambda z}}{1 - e^{-\lambda z}}$

$$= \frac{1 - (\lambda z + \cdots)}{(\lambda z + \cdots)}$$

$\Rightarrow h^*(z) \to \dfrac{1}{\lambda z}$

By def, $h^*(z) = \int_0^\infty e^{-zt} \cdot h(t)\,dt$

$\therefore$ As $z \to 0$, $h^*(z) = \int_0^\infty h(t)\,dt \to \dfrac{1}{\lambda z}$

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$$\Rightarrow \int_0^\infty h(t)\,dt \;\propto\; \frac{1}{\lambda}.$$

Hence as $t \to \infty$, $\boxed{h(t)\,dt \to \frac{1}{\lambda}}$,