Thm: A function $f: E \to R$ is measurable iff the set $\{x \in E ; f(x) > r\}$ is measurable for every rational no. r.

Proof: Let $f: E \to R$ is measurable.

$\therefore \{x \in E \mid f(x) > \alpha\}$ is measurable $\forall \alpha \in R$.

As every rational no is a real no.

$\therefore$ The set $\{x \in E ; f(x) > r\}$ is also measurable for every rational no. r.

Conversely, let $f: E \to R$ be such that the set $\{x \in E ; f(x) > r\}$ is measurable, for every rational no. r.

We shall prove that f is a measurable func.

Let $\alpha \in R$, then $\exists$ a monotonically decreasing sequence $\langle r_n \rangle$ of rational nos. s.t. $\{r_n\} \to \alpha$.

Consider the set $\{x \in E ; f(x) > \alpha\}$

$$= \bigcup_{n=1}^{\infty} \{x \in E ; f(x) > r_n\} \quad - (1)$$

According to the hypothesis, each set on R.H.S of (1) is measurable

& countable union of measurable sets is measurable

$\therefore$ The set of on the R.H.S is measurable.

Hence the set on L.H.S i.e $\{x \in E ; f(x) > \alpha\}$ is measurable.

Hence, f is a measurable function.

Characteristic Function of a set: Let A be a non-empty subset of X on the real line

define $\chi_A : X \to R$ s.t.

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$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in A^c \end{cases}$$

Then χ_A is called characteristic of the set X.

Note : i) If $\chi_A = \chi_B \Rightarrow A = B$.
ii) $\chi_{A^c} = 1 - \chi_A$
iii) $\chi_X = 1$
iv) $\chi_\phi = 0$

Thm: If X is a measurable subset of $\mathbb{R}$, then prove that χ_A is measurable iff A is measurable where $A \subset X$.

Proof: let A be a measurable set and $\alpha \in \mathbb{R}$

Consider the set $\{x \in X ; \chi_A(x) > \alpha\}$

$$= \begin{cases} X & \text{if } \alpha < 0 \\ A & \text{if } 0 \le \alpha < 1 \\ \phi & \text{if } \alpha \ge 1 \end{cases} \quad - (1)$$

All the set X, A, ϕ are measurable

$\therefore$ R.H.S of (1) is measurable

$\therefore$ The set $\{x \in X ; \chi_A(x) > \alpha\}$ is measurable.

Hence χ_A is a measurable func.

Conversely, Suppose the func. χ_A is measurable then the set $\{x \in X ; \chi_A(x) > 0\}$
$= A$

$\because \chi_A$ is a measurable function

$\therefore$ The set $\{x \in X ; \chi_A(x) > 0\}$ is measurable

Hence, A is a measurable set.