Hermite's Polynomials

Hermite's Polynomial is denoted by $H_n(x)$

$$\boxed{H_n(x) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}}$$

Where $\left(\frac{n}{2}\right) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd.} \end{cases}$

Generating function for $H_n(x)$:-

To prove $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$

proof :-

$$\text{L.H.S} = e^{2tx-t^2}$$

$$= e^{2tx} \cdot e^{-t^2}$$

$$= \sum_{r=0}^{\infty} \frac{(2xt)^r}{r!} \left(\sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!} \right)$$

$$\left\{ \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} \right\}$$

we can write it as

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (2x)^r}{r!\, s!} t^{r+2s}$$

$$\left\{ \begin{array}{l} \text{Put } r+2s = n,\ r = n-2s \\ \text{But } r \geq 0,\quad n-2s \geq 0 \\ \qquad\qquad n \geq 2s \\ \qquad\qquad 2s \leq n \\ \qquad\qquad s \leq \frac{n}{2} \end{array} \right\}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{s=0}^{n/2} \frac{(-1)^s (2x)^{n-2s} t^n \cdot n!}{(n-2s)!\, s!} \right) \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}$$

$$\boxed{e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}}$$

V.V. Imp — Rodrigues Formula for $H_n(x)$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

proof :- The generating function for Hermite polynomial $H_n(x)$ is given by

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}$$

$$e^{x^2-x^2+2tx-t^2} = \frac{H_0(x)t^0}{0!} + \frac{H_1(x)t^1}{1!} + H_2(x)t^2 + \cdots + \frac{H_n(x)t^n}{n!} + \frac{H_{n+1}(x)t^{n+1}}{(n+1)!} + \cdots$$

$$\Rightarrow e^{x^2-(x^2+t^2-2tx)} = H_0(x) + \frac{H_1(x)t}{1!} + \frac{H_2(x)t^2}{2!} + \cdots + \frac{H_n(x)t^n}{n!} + \frac{H_{n+1}(x)t^{n+1}}{(n+1)!} + \cdots$$

$$\Rightarrow e^{x^2-(x-t)^2} = H_0(x) + \frac{H_1(x)t}{1!} + \frac{H_2(x)t^2}{2!} + \cdots + \frac{H_n(x)t^n}{n!} + \frac{H_{n+1}(x)t^{n+1}}{(n+1)!} + \cdots$$

Partially differentiating both sides w.r.t 't' n-times

$$\frac{\partial^n}{\partial t^n}\left[e^{x^2-(x-t)^2}\right] = 0+0+0+\cdots + \frac{H_n(x)n!}{n!} + \frac{H_{n+1}(x)(n+1)!\,t}{(n+1)!} + \cdots$$

$$= \frac{\partial^n}{\partial t^n}\left[e^{x^2-(x-t)^2}\right] = \frac{H_n(x)n!}{(n+1)!} + \frac{H_{n+1}(x)(n+1)!\,t}{(n+1)!} + \cdots$$

Put $t=0$ in above eqn, we get

$$\left[\frac{\partial^n}{\partial t^n}\left\{e^{x^2-(x-t)^2}\right\}\right]_{t=0} = \frac{H_n(x)\cancel{n!}}{\cancel{n!}}$$

$$e^{x^2}\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(x-t)^2}\right\}\right] = H_n(x) \quad \text{--- } (A)$$

Put $x-t=u$ --- (*)

Partially diff (*) w.r.t 't' we get

$$0-1 = \frac{\partial u}{\partial t}$$

$$\boxed{\frac{\partial u}{\partial t} = -1}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial t} \Rightarrow \frac{\partial}{\partial t} = -\frac{\partial}{\partial u}$$

$$\boxed{\frac{\partial^n}{\partial t^n} = (-1)^n \frac{\partial^n}{\partial u^n}} \quad \text{--- } (1)$$

$$\left\{\begin{matrix}\text{At } t=0 \\ x = u\end{matrix}\right\} \quad \text{--- } (2)$$

<u>Now</u> using (1), (2) in (A) equation

$$e^{x^2}\left[(-1)^n\frac{\partial^n}{\partial u^n}\left(e^{-u^2}\right)\right]_{t=0} = H_n(x)$$

$$e^{x^2}\left[(-1)^n\frac{\partial^n}{\partial x^n}\left(e^{-x^2}\right)\right] = H_n(x)$$

$$\boxed{\because \text{At } t=0,\ x=u}$$

$$H_n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} \left(e^{-x^2}\right)$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right)$$

[Replace $\frac{\partial^n}{\partial x^n}$ with $\frac{d^n}{dx^n}$]

$$\boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right)}$$