Gram-Schmidt Orthogonalization Process :

Let X be an inner product space. If $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent set in X, then $\exists$ an orthonormal set $\{e_1, e_2, \ldots, e_n\}$ s.t

$$\text{span}\{x_1, x_2, \ldots, x_k\} = \text{span}\{e_1, e_2, \ldots, e_k\}$$

$$\forall\ k = 1, 2, \ldots, n.$$

Proof: As $\{x_1, x_2, \ldots, x_n\}$ is L.I,

$\therefore\ x_1 \neq 0$

Choose $e_1 = \dfrac{x_1}{\|x_1\|}$

$\Rightarrow \|e_1\| = 1.$

and $e_1 \in \text{span}\{x_1\}$

$\Rightarrow \text{span}\{e_1\} \subseteq \text{span}\{x_1\}$

Also $\dim(\text{span}\{e_1\}) = 1 = \dim(\text{span}\{x_1\})$

$\Rightarrow \text{span}\{e_1\} = \text{span}\{x_1\}$

Now, if $y_2 = x_2 - \langle x, e_1 \rangle e_1$.

Then $\langle y_2, e_1 \rangle = 0$ [Show it]

Also $y_2 \neq 0$

$\because$ if $y_2 = 0$

Then $x_2 \in \text{span}\{e_1\} = \text{span}\{x_1\}$

$\Rightarrow \{x_1, x_2\}$ is L.D

which is not so.

$\therefore$ ~~x~~ $y_2 \neq 0$.

and if $e_2 = \dfrac{y_2}{\|y_2\|}$

Then $\|e_2\| = 1$

and $\langle e_1, e_2 \rangle = \left\langle e_1, \dfrac{y_2}{\|y_2\|} \right\rangle$

$$= \frac{1}{\|y_2\|} \langle e_1, y_2 \rangle$$

$$= 0$$

$\Rightarrow \{e_1, e_2\}$ is orthonormal

$\Rightarrow \{e_1, e_2\}$ is L.I.

Also $e_2 \in \text{span}\{y_2\} \subseteq \text{span}\{x_2, e_1\}$

$$= \text{span}\{x_2, x_1\}$$

$\therefore$ $e_1, e_2 \in \text{span}\{x_1, x_2\}$

$\text{span}\{e_1, e_2\} \subseteq \text{span}\{x_1, x_2\}$

Also $\dim(\text{span}\{e_1, e_2\}) = 2 = \dim(\text{span}\{x_1, x_2\})$

$\Rightarrow \text{span}\{e_1, e_2\} = \text{span}\{x_1, x_2\}$.

Let $y_3 = x_3 - \langle x_3, e_1 \rangle e_1 - \langle x_3, e_2 \rangle e_2$

Then $\langle y_3, e_1 \rangle = 0 = \langle y_3, e_2 \rangle$. [Show it]

Also $y_3 \neq 0$

[$\because$ if $y_3 = 0$ then $x_3 \in \text{span}\{e_1, e_2\} = \text{span}\{x_1, x_2\}$
$\Rightarrow \{x_1, x_2, x_3\}$ is L.D
which is not so

$\Rightarrow y_3 \neq 0$

Choose $e_3 = \dfrac{y_3}{\|y_3\|}$

Then $\|e_3\| = 1$

and $\langle e_3, e_1 \rangle = \left\langle \dfrac{y_3}{\|y_3\|}, e_1 \right\rangle = \dfrac{1}{\|y_3\|} \langle y_3, e_1 \rangle$
$= 0$

and $\langle e_3, e_2 \rangle = \left\langle \dfrac{y_3}{\|y_3\|}, e_2 \right\rangle = \dfrac{1}{\|y_3\|} \langle y_3, e_2 \rangle$
$= 0$

$\Rightarrow \{e_1, e_2, e_3\}$ is orthonormal

$\Rightarrow \{e_1, e_2, e_3\}$ is L.I

And $e_1, e_2 \in \text{span}\{x_1, x_2\} \subseteq \text{span}\{x_1, x_2, x_3\}$
and $e_3 \in \text{span}\{y_3\} \subseteq \text{span}\{x_3, e_1, e_2\} = \text{span}\{x_3, x_2, x_1\}$

$\Rightarrow e_1, e_2, e_3 \in \text{span}\{x_1, x_2, x_3\}$

$\Rightarrow \text{span}\{e_1, e_2, e_3\} \subseteq \text{span}\{x_1, x_2, x_3\}$

and as $\{e_1, e_2, e_3\}$ is $\ell \cdot I$

$\Rightarrow \dim(\text{span}\{e_1, e_2, e_3\}) = 3 = \dim(\text{span}\{x_1, x_2, x_3\})$

$\Rightarrow \text{span}\{e_1, e_2, e_3\} = \text{span}\{x_1, x_2, x_3\}$

Continuing like this if $e_1, e_2, -, e_K$ are chosen s.t

$\{e_1, e_2, -, e_K\}$ is orthonormal

and $\text{span}\{e_1, --, e_i\} = \text{span}\{x_1, .., x_i\}$

$\forall\ i \leq K$

Then choose

$$y_{K+1} = x_{K+1} - \langle x_{K+1}, e_1 \rangle e_1 \cdots - \langle x_{K+1}, e_K \rangle e_K.$$

Then as above $y_{K+1} \neq 0$

and $\langle y_{K+1}, e_1 \rangle = \cdots = \langle y_{K+1}, e_K \rangle = 0$

and if $e_{K+1} = \dfrac{y_{K+1}}{\|y_{K+1}\|}$

Then $\|e_{K+1}\| = 1$ and $\langle e_{K+1}, e_1 \rangle = 0$

$\langle e_{K+1}, e_2 \rangle = 0$

$\vdots$

$\langle e_{K+1}, e_K \rangle = 0.$

$\Rightarrow$ ~~set~~ $\{e_1, e_2, \ldots, e_k, e_{k+1}\}$ orthonormal

and hence L.I

Also $e_{k+1} \in \text{span}\{x_{k+1}, e_1, \ldots, e_k\}$

$$= \text{span}\{x_{k+1}, x_k, \ldots, x_1\}$$

$\Rightarrow$ $e_1, \ldots, e_k, e_{k+1} \in \text{span}\{x_1, x_2, \ldots, x_{k+1}\}$

$\Rightarrow$ $\text{span}\{e_1, \ldots, e_k, e_{k+1}\} \subseteq \text{span}\{x_1, \ldots, x_k, x_{k+1}\}$

and $\dim(\text{span}\{e_1, \ldots, e_k, e_{k+1}\}) \overset{=k+1}{=} \dim(\text{span}\{x_1, \ldots, x_k, x_{k+1}\})$

$\Rightarrow$ $\text{span}\{e_1, \ldots, e_k, e_{k+1}\} = \text{span}\{x_1, \ldots, x_k, x_{k+1}\}$

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