8. **LATTICE** :- A Poset $(A, \leq)$ is called lattice if every pair of elements a and b in A has both l.u.b and g.l.b.

$l.u.b\{a, b\} = a \vee b$ (Join of a & b)

$g.l.b\{a, b\} = a \wedge b$ (meet of a & b)

i) $Sup\{a, b\} = a \vee b$
(l.u.b) or $a \cup b$
or $a + b$

ii) $g.l.b\{a, b\} = a \wedge b$
$Inf\{a, b\}$ or $a \cap b$
or $a \cdot b$

Relation	l.u.b	g.l.b
Usual $\leq$	Greater Element	smaller Element
Divide /	l.c.m	g.c.d
Subset $\subseteq$	$A \cup B$	$A \cap B$

Examples :- 1.

$A = \{a, b, c, d\}$

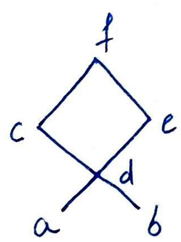
$B = \{a, c\}$

g.l.b of B = a

l.u.b of B = c

$\therefore$ It is a lattice.

2.



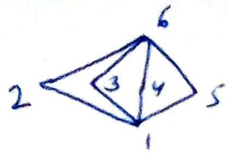
$B = \{a, b\}$

g.l.b = None ($\because$ a and b are on same position and it has no element in its lower side)

$\therefore$ It is not lattice

23.
17.

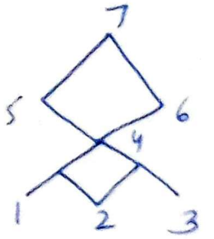
3.



l.u.b = 6 It is a Lattice.

g.l.b = 1

4.



1, 2 & 3 are in same Position & it has no g.l.b ∴ it has no element in lower side.

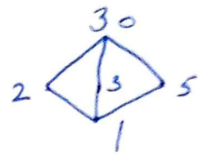
g.l.b = None

It is not lattice.

5. Write operation Table for $\vee$ & $\wedge$ for $L = \{1, 2, 3, 5, 30\}$ under divisibility relation.

Sol:- $a \vee b$ = l.u.b $\{a, b\}$ = l.cm

$a \wedge b$ = g.l.b $\{a, b\}$ = g.c.d



$\vee$	1	2	3	5	30
1	1	2	3	5	30
2	2	2	6	10	30
3	3	6	3	15	30
5	5	10	15	5	30
30	30	30	30	30	30

$\wedge$	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

In $a \vee b$ 6, 10 & 15 are $\notin L$.

∴ It is not Lattice.

6. Write operation Table for $\vee$ & $\wedge$ for $L = \{\phi, \{1\}, \{2\}, \{1,2\}\}$ under $\subseteq$ relation.

Sol: $a \vee b$ = l.u.b $\{a, b\}$ = $a \cup b$

$a \wedge b$ = g.l.b $\{a, b\}$ = $a \cap b$

$\vee$	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
ϕ	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$

$\wedge$	$\{\phi\}$	$\{1\}$	$\{2\}$	$\{1,2\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{1\}$	ϕ	$\{1\}$	ϕ	$\{1\}$
$\{2\}$	ϕ	ϕ	$\{2\}$	$\{2\}$
$\{1,2\}$	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$

All Elements in table are $\in L$

∴ It is a Lattice.

☆ Prove that Every chain is a Lattice.

Sol: Let $\{a, b\}$ be two arbitrary Element of Chain

$a \leq b$

$g.l.b \{a, b\} = a$

$l.u.b \{a, b\} = b$

Hence it is Lattice.

[Chain: A Poset is called chain if Every two Elements are Comparable.

Comparable Elements if a & b are two Elements either $a \leq b$ or $b \leq a$]

☆ S.T. A Lattice with three or fewer Elements is chain.

Sol: Let L be any Lattice:

Case I If $L = \{a\}$ i.e. one Element

$\cdot a$ It is obvious that itself Comparable.

$\therefore$ It is chain.

Case II If $L = \{a, b\}$ $l.u.b\ L = \{b\}$

$g.l.b\ L = \{a\}$

$\therefore a \leq b$

$\therefore$ comparable

$\therefore$ It is chain

Case III : If $L = \{a, b, c\}$

$l.u.b = \{C\}$ $\therefore a \leq c$

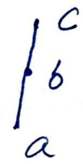
$g.l.b = \{a\}$

$(a, b) \quad (b, c) \quad (a, c)$

$a \wedge b = g.l.b \{a, b\} = a$

$\therefore a \leq b$

$b \vee c = l.u.b \{b, c\} = c$



$\therefore b \leq c$

$\therefore \quad a \leq b, \; b \leq c$ and $a \leq c$

these all Elements are Comparable

$\therefore$ It is chain.

Let $(L, \leq)$ be a Lattice. for any Elements $a, b \in L$

P.T. $a \wedge b = a$ iff $a \vee b = b$

Sol: Let $a \wedge b = a$ T.P $a \vee b = b$

$g.l.b \{a, b\} = a$

$\therefore a \vee b = l.u.b \{a, b\}$

$= b$ Hence $a \vee b = b$

Converse :- Let $a \vee b = b$

$l.u.b \{a, b\} = b$

$\therefore a \wedge b = g.l.b \{a, b\} = a$

Hence $a \wedge b = a$

Hence Proved

Thm: Let P be a Poset under '$\leq$' & $x, y \in P$

then i) If x, y have l.u.b then l.u.b is unique.

ii) If x, y have g.l.b then this g.l.b is unique.

Proof: i) If Possible suppose l_1, l_2 be two distinct

l.u.b of x, y.

Let l_1 be l.u.b & $l_2 \in P$

$$l_2 \leq l_1 \quad - (1)$$

Let l_2 be l.u.b & $l_1 \in P$

$$l_1 \leq l_2 \quad - (2)$$

By (1) & (2) $l_1 = l_2$

ii) ~~Let~~ If Possible suppose u_1, u_2 be two distinct g.l.b of x, y.

Let u_1 is g.l.b & $u_2 \in P$

$$u_1 \leq u_2 \quad - (1)$$

Let u_2 is g.l.b & $u_1 \in P$

$$u_2 \leq u_1 \quad - (2)$$

By (1) & (2)

$$u_1 = u_2$$

Hence Proved.

9 Product of Lattice :- Let A & B be two Lattice.

$A \times B = \{(a, b) \mid a \in A \ \& \ b \in B\}$ is a Poset under relation '$\leq$' defined by $(a_1, b_1) \leq (a_2, b_2)$ iff $a_1 \leq a_2$ in A & $b_1 \leq b_2$ in B.

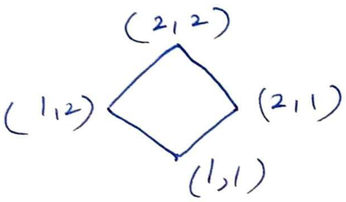
$A \times B$ is called Product of Lattice.

example :- Consider $(L, \leq)$ where $L = \{1, 2\}$ find $(L^2, \leq)$ Also draw Hasse Diagram.

Sol : $L \times L = \{1, 2\} \times \{1, 2\}$

$= \{(1,1), (1,2), (2,1), (2,2)\}$

$(1,1) \leq (1,2)$ iff $1 \leq 1, \ 1 \leq 2$.



Thm : P.T. Product of Two Lattice is a Lattice.

Proof : Let L & M be Two Lattice. Then $L \times M = \{(l, m) \mid l \in L, m \in M\}$ is defined by $(l_1, m_1) \leq (l_2, m_2)$ iff $l_1 \leq l_2$ & $m_1 \leq m_2$.

T.P $L \times M$ is a Lattice.

Let $(l_1, m_1), (l_2, m_2) \in L \times M$

$l_1, l_2 \in L$; $m_1, m_2 \in M$.

Since L, M are Lattice

$\therefore l_1 \wedge l_2 \in L$ & $m_1 \wedge m_2 \in M$

$\Rightarrow l_1 \wedge l_2 \leq l_1$ & $m_1 \wedge m_2 \leq m_1$

& $l_1 \wedge l_2 \leq l_2$ & $m_1 \wedge m_2 \leq m_2$

$\Rightarrow (l_1 \wedge l_2, m_1 \wedge m_2) \leq (l_1, m_1)$

& $(l_1 \wedge l_2, m_1 \wedge m_2) \leq (l_2, m_2)$

$\therefore (l_1 \wedge l_2, m_1 \wedge m_2)$ is lower bound of

$\{(l_1, m_1), (l_2, m_2)\}$

Now Suppose (l, m) be other lower bound of $\{(l_1, m_1), (l_2, m_2)\}$

$\Rightarrow (l, m) \leq (l_1, m_1)$

& $(l, m) \leq (l_2, m_2)$

$\Rightarrow l \leq l_1, l \leq l_2$ & $m \leq m_1, m \leq m_2$

$\Rightarrow l$ is lower bound of $\{l_1, l_2\}$ in L.

& m is lower bound of $\{m_1, m_2\}$ in M.

$\Rightarrow l \leq l_1 \wedge l_2$ ($\because$ lower bound $\leq$ g.l.b)

& $m \leq m_1 \wedge m_2$

$\Rightarrow (l, m) \leq (l_1 \wedge l_2, m_1 \wedge m_2)$

$\therefore (l_1 \wedge l_2, m_1 \wedge m_2)$ is g.l.b of $\{(l_1, m_1), (l_2, m_2)\}$

Similarly We can show that $\{l_1 \vee l_2, m_1 \vee m_2\}$ is l.u.b of $\{(l_1, m_1), (l_2, m_2)\}$.

Thus both l.u.b & g.l.b Exists.

Hence $L \times M$ is a lattice.